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Critical exponent ν of the self-avoiding walks on plane X and checkerboard fractal families

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Abstract. We have studied the critical exponent ν associated with the mean squared end-to-end distance of self-avoiding walks on the two-dimensional X and checkerborard fractal families. By means of exact renormalization group transformations we have calculated ν for the first four members of each family.

In recent years many specific results have demonstrated that the critical behaviour of statistical model systems on fractals depends on underlying lattice structures. This is particularly true in the case of the self-avoiding walks (SAWs), where so far no exact formula that represents critical exponents in terms of fractal and spectral dimensions and other fractal properties has been found. However, there are many phenomenological proposals for such a formula, but their validity has not been supported by exact results for SAWs on the Sierpinski gasket family of fractals [1, 2]. For this reason, additional exact results on different fractal families are desirable. In this paper we present values of the mean squared end-to-end distance critical exponent ν for self-avoiding walks on plane X and checkerboard (CB) fractal families.

Each member of the plane X and CB family is labelled by an odd integer $b=2k+1$ ($k=1,2,\ldots,\infty$) and can be obtained as the result of an infinite iterative process of successively enlarging the fractal structure b times and substituting the smallest parts of the enlarged structure with the generator (initial structure). The generator of a CB fractal is a square, of size $b\times b$, composed of b rows of unit squares, so that within each row and each column every other one of them is removed, whereas in the case of X fractals instead of unit squares we put crosses composed of the diagonals of squares (see figure 1). The fractal dimension d_f for an arbitrary member specified by b, of each of the two famiilies, is equal to $\ln[(b^2+1)/2]/\ln b$ [3], whereas the spectral dimension d_s is known only for the few first members of both families [4]. For increasing b, one can expect that members of these two families become more and more similar to the ordinary square lattice, which is indicated by the fact that the fractal dimension d_f tends to 2 when $b\to\infty$. (The $b=\infty$ fractal generator is a $\pi/2$ wedge of the square lattice, for both families.)

The SAW model represents a random walk that must not contain self-intersections. Its mean squared end-to-end distance $\langle R_N^2\rangle$, for N steps, should behave according to the scaling relation

$$\langle R_N^2\rangle \sim N^{2\nu} \tag{1}$$

in the asymptotic region $N\to\infty$. In our study of SAWs we apply the real-space renormalization group (RG) technique [5] to calculate the critical exponent ν. To this end, let

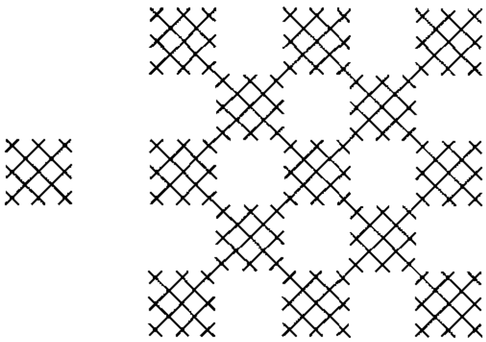

Figure 1. The first two steps of construction of the $b=5$ plane X fractal lattice.

us consider any finite part of the CB, or X, fractal lattice that is the same as the structure obtained after the rth step of the iterative construction of the entire fractal, and let us call it the rth-order generator. Next, we introduce the three restricted partition functions

$$F^{(r)}(x)=\sum_n F_n x^n \qquad G^{(r)}(x)=\sum_n G_n x^n \qquad H^{(r)}(x)=\sum_n H_n x^n \tag{2}$$

where x^n is the weight of an n-step SAW, while F_n is the number of all possible n-step SAWs that start at one corner and end at the nearest corner of an rth-order generator. Furthermore, G_n in (2) is the number of n-step SAWs that start at one corner and end at the diagonally opposite corner of the rth-order generator, and H_n is the number of n-step SAWs that consist of two non-intersecting SAW parts, such that each of them starts at one corner and ends at one of two nearest corners of the rth-order generator (see figure 2). For any $b>3$ X fractal, the initial values of these functions that

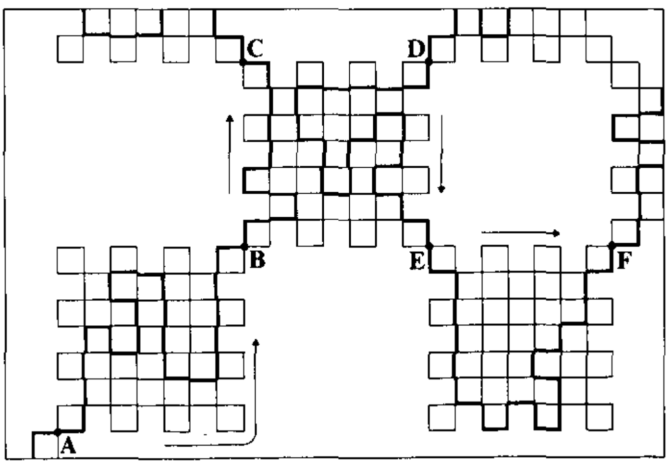

Figure 2. An illustration of the restricted partition functions used to describe SAWs on fractals. Here we focus our attention on the SAW from the vertex A to the vertex F which is a part of a longer SAW somewhere within by $b=7$ CB fractal. The SAW part from A to B is of the G type and is weighted by x^{26} within the restricted partition function $G^{(1)}(x)$ (see equation (2)). Similarly, the BC part, together with the DE part, contributes a term with the weight x^{38} in $H^{(1)}(x)$, while the EF part of the SAW is weighted by x^{25} and makes a contribution to the function $F^{(1)}(x)$.

correspond to a unit cross are given by

$$F^{(0)}=x^2 \qquad G^{(0)}=x^2 \qquad H^{(0)}=0 \tag{3}$$

whereas for every particular CB-fractal one has to start with polynomials in x that correspond to the fractal generator (that is, not to a unit square) and whose order and coefficients depend on b† (see appendix 1). Since the fractals are self-similar structures, recursion relations that connect the $(r+1)$th and rth-order restricted partition functions are independent of r, and it can be verified that (for general b) they have the following form

$$\begin{aligned} F^{(r+1)} &= \sum_{i,j,k} A_b(i,j,k) F^{(r)^i} G^{(r)^j} H^{(r)^k} \\ G^{(r+1)} &= \sum_{i,j,k} B_b(i,j,k) F^{(r)^i} G^{(r)^j} H^{(r)^k} \\ H^{(r+1)} &= \sum_{i,j,k} C_b(i,j,k) F^{(r)^i} G^{(r)^j} H^{(r)^k}. \end{aligned} \tag{4}$$

One should observe that the polynomials given in (4) are valid for both the X and CB fractal, with the same b, and can be obtained by summing all possible self-avoiding paths on the CB fractal generator. The latter paths consist of the edges and diagonals of the unit squares, with the restriction that, within a square, two neighbouring edges cannot be traversed subsequently. One illustrating example is given in figure 3.

For $b=3$ and $b=5$ we have determined coefficients that appear in relations (4) by a straightforward calculation, whereas for $b=7$ and $b=9$ the work had to be computerized. The direct enumeration of all relevant paths for $b\geqslant 11$ is infeasible using a PC with the Intel 80486 processor, and hardly possible using a mainframe comparable with the IBM 3090. In appendix 2 we present coefficients A, B and C for $b=3, 5, 7$ (and the CPU time required to obtain them), but we do not give our data for $b=9$ as

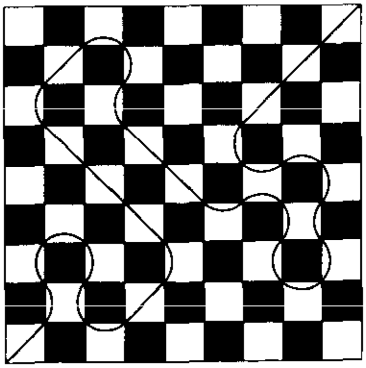

Figure 3. An example of the SAW path that contributes to the second recursion relation (4), in the case of the $b=9$ fractal. The corresponding weight of the path is $F^{(r)^{16}}G^{(r)^{11}}H^{(r)^3}$, whereas the total number of paths with the same weight is $B_9(16, 11, 3) = 245\,556$. In this picture, each white square represents an rth-order generator.

† Only never-starting and never-ending SAWs are treated in the RG approach applied in this paper. It is easy to see that on the $b=3$ fractal (X or CB) infinite SAWs can be performed only by making G-type walks, that is, the F and H partition functions are not relevant, so that we may put $F^{(r)}(x)=H^{(r)}(x)=0$ and proceed further.

the corresponding presentation would require ten printed journal pages (the data are available upon request addressed to the authors).

Partition functions F, G and H are the RG parameters and relations (4) represent the corresponding exact RG transformations. A fixed point (F^*, G^*, H^*) of these transformations can be found by numerically solving the nonlinear system of equations

$$\begin{aligned} F^* &= \sum_{i,j,k} A_b(i,j,k)(F^*)^i(G^*)^j(H^*)^k \\ G^* &= \sum_{i,j,k} B_b(i,j,k)(F^*)^i(G^*)^j(H^*)^k \\ H^* &= \sum_{i,j,k} C_b(i,j,k)(F^*)^i(G^*)^j(H^*)^k. \end{aligned} \tag{5}$$

For each fractal studied here, specified by a particular b, there is more than one non-trivial fixed point. A numerical study of the flow in the (F, G, H) space showed which one of the fixed points is relevant to the determination of the asymptotic behaviour of SAWs on a fractal. It turns out that the linearized RG transformations, at the relevant fixed point, have only one eigenvalue $\lambda(b)$ larger than 1, so that one can straightforwardly determine the critical exponent ν according [5] to the formula

$$\nu(b) = \ln b / \ln \lambda(b). \tag{6}$$

Specific results for the fixed points and the relevant eigenvalues for the fractals under study are given in appendix B. The pertinent critical exponent ν values are 1, 0.852 35, 0.815 02 and 0.795 78, for b equal to 3, 5, 7 and 9, respectively.

The results obtained, plotted versus the variable $1/b$, are depicted in figure 4. One can see that ν monotonically decreases with increasing b, and, as in the case of the Sierpinski gasket (SG) family of fractals [1], one could expect that ν continues to

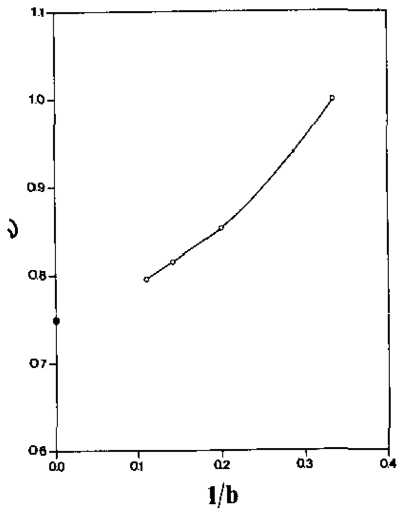

Figure 4. The exact results (○) for the critical exponent ν as a function of $1/b$, where b enumerates members of the fractal families under study. The full curve that connects the presented results serves merely as a guide to the eye. The solid circle (●) represents the value $\nu = \frac{3}{4}$ found for the Euclidean lattices [6].

approach monotonically the Euclidean value $\nu = \frac{3}{4}$ [6], when $b \to \infty$. However, in view of the conclusions obtained by the finite-size scaling arguments [7], and supported by the Monte Carlo RG results [2], one should be cautious in advocating such an expectation. Indeed, it has been shown [2, 7], in the case of the SG fractals, that ν can cross the Euclidean value for some finite b, that is, before the crossover region at $b \to \infty$. Accepting appropriate finite-size scaling assumptions, we can analyse our results, in the way Dhar [7] did for the SG fractals, so as to reach a conclusion that is similar to the one reached in the SG case. Unfortunately, the set of existing numerical data cannot yet offer necessary support to the prerequisite scaling assumptions. Therefore, we can conclude that the study of SAWs on the X and CB fractals brings about the same questions posed in the case of the SG fractals and, hence, the problem under study, although more complex, merits further investigation.

Acknowledgments

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Appendix 1

Here we present the initial values (i.e. initial polynomials in x) for the restricted partition functions. In the case $b=3$, the only relevant parameter G has the initial values $G^{(0)} = x^2$ and $G^{(0)} = 2x^2$ for the X and CB fractals, respectively. The respective critical values of x, at which $G^{(r)}(x)$ starts to diverge, are $x^* = 1$ and $x = \sqrt{\frac{1}{2}}$.

In the case of X fractals, with $b > 3$, there are unique initial values given by (3), whereas for CB fractals, with $b > 3$, accepting $F^{(0)}$, $G^{(0)}$ and $H^{(0)}$ as initial values could imply self-intersections for walks described by the higher-order restricted partition functions. For this reason, for $b = 5$, as the initial conditions we take the first-order restricted partition functions

$$\begin{aligned} F^{(1)}(x) = {} & 4x^7 + 28x^9 + 72x^{11} + 216x^{13} + 520x^{15} + 820x^{17} + 876x^{19} + 644x^{21} + 312x^{23} \\ & + 84x^{25} + 8x^{27} \end{aligned}$$

$$G^{(1)}(x) = 80x^{10} + 208x^{12} + 400x^{14} + 592x^{16} + 784x^{18} + 752x^{20} + 464x^{22} + 176x^{24} + 32x^{26}$$

$$\begin{aligned} H^{(1)}(x) = {} & 16x^{14} + 224x^{16} + 1\,360x^{18} + 2\,176x^{20} + 2\,512x^{22} + 2\,112x^{24} + 1\,360x^{26} + 608x^{28} \\ & + 128x^{30} \end{aligned}$$

while for $b = 7$ we take

$$\begin{aligned} F^{(1)}(x) = {} & 4x^9 + 68x^{11} + 400x^{13} + 1\,656x^{15} + 6\,496x^{17} + 25\,500x^{19} + 97\,268x^{21} + 328\,924x^{23} \\ & + 950\,448x^{25} + 2\,360\,964x^{27} + 5\,076\,780x^{29} + 9\,418\,916x^{31} + 14\,947\,836x^{33} \\ & + 20\,107\,100x^{35} + 22\,725\,912x^{37} + 21\,383\,764x^{39} + 16\,574\,216x^{41} \\ & + 10\,443\,260x^{43} + 5\,249\,696x^{45} + 2\,043\,264x^{47} + 585\,744x^{49} + 113\,404x^{51} \\ & + 12\,528x^{53} + 508x^{55} \end{aligned}$$

$$
\begin{aligned}
G^{(1)}(x) = {} & 1\,008x^{14} + 5\,792x^{16} + 22\,336x^{18} + 70\,848x^{20} + 201\,744x^{22} + 535\,584x^{24} \\
& + 1\,340\,896x^{26} + 3\,124\,992x^{28} + 6\,567\,472x^{30} + 11\,954\,464x^{32} \\
& + 18\,265\,376x^{34} + 23\,068\,096x^{36} + 23\,967\,120x^{38} + 20\,435\,472x^{40} \\
& + 14\,213\,808x^{42} + 7\,961\,760x^{44} + 3\,510\,896x^{46} + 1\,173\,712x^{48} + 279\,248x^{50} \\
& + 42\,080x^{52} + 3\,024x^{54}
\end{aligned}
$$

$$
\begin{aligned}
H^{(1)}(x) = {} & 16x^{18} + 544x^{20} + 7\,824x^{22} + 67\,648x^{24} + 437\,184x^{26} + 1\,810\,784x^{28} \\
& + 5\,478\,624x^{30} + 13\,110\,432x^{32} + 26\,000\,288x^{34} + 43\,668\,544x^{36} \\
& + 62\,477\,552x^{38} + 75\,790\,592x^{40} + 77\,226\,336x^{42} + 65\,469\,376x^{44} \\
& + 45\,795\,728x^{46} + 26\,185\,984x^{48} + 12\,058\,816x^{50} + 4\,353\,408x^{52} \\
& + 1\,169\,936x^{54} + 209\,696x^{56} + 18\,800x^{58}.
\end{aligned}
$$

The corresponding critical values of x^* are $x^*_{\mathrm{CB}}(b=5)=0.5648$, $x^*_{\mathrm{CB}}(b=7)=0.5102$, $x^*_{\mathrm{X}}(b=5)=0.8359$ and $x^*_{\mathrm{X}}(b=7)=0.7715$. Here we would like to point out that, roughly speaking, determination of $F^{(1)}$, $G^{(1)}$ and $H^{(1)}$ using a PC with the Intel 80486 processor required a few minutes for the $b=5$ CB fractal, and a few hours for the $b=7$ CB fractal, so that the $b=9$ case was not accessible. Fortunately, our analysis of the RG equations (4) for $b=3$, 5, and 7 confirms that X and CB fractals have the same fixed points. Hence, for the $b=9$ CB fractal we have accepted the fixed point values found for the $b=9$ X fractal (see appendix 2), which was determined in accord with the corresponding critical value $x^*_{\mathrm{X}}(b=9)=0.7355$.

Appendix 2

In this appendix we present coefficients $A_b(i,j,k)$, $B_b(i,j,k)$ and $C_b(i,j,k)$ of the RG transformations (4) for b, 3, 5, and 7.

$b=3$:

$A_3(i,j,k)=0$ $\quad B_3(0,3,0)=1$ $\quad C_3(i,j,k)=0$

$b=5$:

$A_5(1,4,0)=1$	$A_5(3,2,0)=1$	$A_5(3,4,0)=2$
$A_5(3,4,1)=1$	$A_5(5,2,1)=2$	$A_5(5,2,3)=3$
$A_5(5,4,0)=3$	$A_5(5,4,1)=2$	$A_5(5,4,2)=2$
$A_5(7,2,0)=1$	$B_5(0,5,0)=1$	$B_5(4,3,0)=6$
$B_5(4,3,2)=4$	$B_5(4,5,2)=2$	$B_5(6,3,0)=2$
$B_5(6,3,1)=4$	$C_5(0,8,1)=1$	$C_5(4,4,5)=8$
$C_5(4,6,0)=2$	$C_5(4,6,1)=4$	$C_5(6,4,0)=1$
$C_5(6,4,2)=6$		

$b=7$:

$A_7(1,6,0)=1 \qquad A_7(3,4,0)=3$

$\{A_7(3,6,i)\}=\{4,1\}$ $\quad i=0,1$

$\{A_7(3,8,i)\}=\{2,2\}$ $\quad i=0,1$

$A_7(5,2,0)=1$

$\{A_7(5,4,i)\}=\{8,6,4,3\}$

$\{A_7(5,6,i)\}=\{12,16,4,12\}$

$\{A_7(5,8,i)\}=\{14,13,10,9\}$ $\quad i=0,\ldots,3$

$\{A_7(5,10,i)\}=\{2,4\}$ $\quad i=1,2$

$\{A_7(7,2,2i+1)\}=\{3,6,8\}$ $\quad i=0,\ldots,2$

$\{A_7(7,4,i)\}=\{17,18,26,20,16,14,11\}$ $\quad i=0,\ldots,2$

$\{A_7(7,6,i)\}=\{44,57,50,49,36,30,22)\}$ $\quad i=0,\ldots,6$

$\{A_7(7,8,i)\}=\{28,44,50,36,19,16\}$ $\quad i=0,\ldots,5$

$\{A_7(7,10,i)\}=\{6,4\}$ $\quad i=1,2$

$\{A_7(9,2,2i)\}=\{3,8,12,46,58,95\}$ $\quad i=0,\ldots,5$

$\{A_7(9,4,i)\}=\{34,42,66,63,80,74,94,42,58,132\}$ $\quad i=0,\ldots,9$

$\{A_7(9,6,i)\}=\{85,124,154,146,126,134,33,36,32\}$ $\quad i=0,\ldots,8$

$\{A_7(9,8,i)\}=\{28,62,76,67,54,16\}$ $\quad i=0,\ldots,5$

$A_7(9,10,1)=2$

$\{A_7(11,2,2i+1)\}=\{7,30,40,68\}$ $\quad i=0,\ldots 3$

$\{A_7(11,4,i)\}=\{58,72,136,126,182,138,177,74,45\}$ $\quad i=0,\ldots,8$

$\{A_7(11,6,i)\}=\{84,145,200,164,142,61,26\}$ $\quad i=0,\ldots,6$

$\{A_7(11,8,i)\}=\{8,24,27,6\}$

$\{A_7(13,2,2i)\}=\{5,15,36,32\}$ $\quad i=0,\ldots\ 3$

$\{A_7(13,4,i)\}=\{40,102,122,164,120,122\}$ $\quad i=0,\ldots,5$

$\{A_7(13,6,i)\}=\{42,90,88,62,13\}$ $\quad i=0,\ldots,4$

$\{A_7(15,2,i)\}=\{6,11\}$ $\quad i=1,3$

$\{A_7(15,4,i)\}=\{18,32,40,12,7\}$ $\quad i=0,\ldots,4$

$B_7(0,7,0)=1$

$\{B_7(4,5,i)\}=\{20,8\}$

$\{B_7(4,7,i)\}=\{6,14\}$ $\quad i=0,2$

$B_7(4,9,2)=12,\ B_7(4,11,2)=4$

$\{B_7(6,5,i)\}=\{36,36,16\}$ $\quad i=0,1,3$

$\{B_7(6,7,i)\}=\{20,32,24,24\}$ $\quad i=0,\ldots,3$

$\{B_7(6,9,i)\}=\{2,12,32\}$ $\quad i=1,2$

$\{B_7(8,3,2i)\}=\{30,48,72,48,72\}$ $\quad i=0,\ldots,4$

$\{B_7(8,5,i)\}=\{54\ 84,90,60,108,32,120,96,72\}$

$\{B_7(8,7,i)\}=\{36,112,108,76,44,32,32,0,36\}$ $\quad i=0,\ldots,8$

$\{B_7(8,9,i)\}=\{8,36,44,0,4\}$ $\quad i=0,\ldots,4$

$\{B_7(10,3,i)\}=\{20,64,24,104,20,144,0,72\}$

$\{B_7(10,5,i)\}=\{60,144,146,212,198,188,172,216\}$ $\quad i=0,\ldots,7$

$\{B_7(10,7,i)\}=\{40,160,172,144,52,52\}$ $\quad i=0,\ldots,5$

$\{B_7(10,9,i)\}=\{8,12,4\}$ $\quad i=0,1,2,$

$\{B_7(12,3,i)\}=\{8,44,72,60,112,68,100\}$

$\{B_7(12,5,i)\}=\{48,140,184,208,244,120,76\}$ $\quad i=0,\ldots,6$

$\{B_7(12,7,i)\}=\{24,76,76,12\}$ $\quad i=0,\ldots,3$

$\{B_7(14,3,i)\}=\{4,20,36,64,16,24\}$ $\quad i=0,\ldots,5$

$\{B_7(14,5,i)\}=\{34,76,82,72,10\}$ $\quad i=0,\ldots,4$

$\{B_7(16,3,i)\}=\{6,14,16\}$ $\quad i=0,1,2$

$C_7(0, 12, 1) = 1 \qquad C_7(4, 8, 5) = 8$

$\{C_7(4, 10, i)\} = \{6, 24, 0, 8, 0, 32\} \qquad i = 0, \ldots, 5$

$\{C_7(4, 12, i)\} = \{4, 16\} \qquad i = 1, 5$

$\{C_7(6, 8, 2i)\} = \{11, 30, 64, 40\} \qquad i = 0, \ldots, 3$

$\{C_7(6, 10, i)\} = \{16, 28, 40, 64, 32\} \qquad i = 0, \ldots, 5$

$C_7(6, 12, 1) = 8 \qquad C_7(8, 4, 13) = 576$

$\{C_7(8, 6, i)\} = \{6, 0, 0, 0, 0, 32, 0, 160, 0, 200\} \qquad i = 0, \ldots, 9$

$\{C_7(8, 8, i)\} = \{56, 144, 132, 228, 156, 158, 96, 72\} \qquad i = 0, \ldots, 7$

$\{C_7(8, 10, i)\} = \{14, 32, 70, 72, 128, 64\} \qquad i = 0, \ldots, 5$

$C_7(8, 12, 1) = 4$

$\{C_7(10, 4, i)\} = \{1, 150, 760\} \qquad i = 0, 8, 10$

$\{C_7(10, 6, i)\} = \{16, 30, 68, 58, 212, 92, 312, 206, 264\} \qquad i = 0, \ldots, 8$

$\{C_7(10, 8, i)\} = \{66, 188, 267, 344, 274, 144, 96\} \qquad i = 0, \ldots, 6$

$\{C_7(10, 10, i)\} = \{4, 20, 40, 10\} \qquad i = 0, \ldots, 4$

$\{C_7(12, 4, 2i+1)\} = \{6, 24, 168, 456, 190\} \qquad i = 0, \ldots, 4$

$\{C_7(12, 6, i)\} = \{52, 116, 158, 272, 252, 340, 72, 72\} \qquad i = 0, \ldots, 7$

$\{C_7(12, 8, i)\} = \{28, 82, 100, 134, 64, 8\} \qquad i = 0, \ldots, 5$

$\{C_7(14, 4, 2i)\} = \{6, 58, 152, 135\} \qquad i = 0, \ldots, 3$

$\{C_7(14, 6, i)\} = \{24, 62, 116, 80, 64, 22\} \qquad i = 0, \ldots, 5$

$\{C_7(16, 4, i)\} = \{6, 14\} \qquad i = 1, 3$

Calculation of these coefficients required a few seconds and a few minutes for the $b = 5$ and $b = 7$ fractals, respectively, using a PC with the Intel 80486 processor. We have also performed similar calculations in the $b = 9$ fractal case, which took a few hours using the same computer. However, presentation of the corresponding data would take ten journal pages and we do not give it here.

Using the known coefficients in the fixed point equations (5) and solving them numerically we have learnt that the relevant fixed points (F^*, G^*, H^*) are $(-, 1, -)$, $(0.663\,71, 0.724\,64, 0.100\,03)$, $(0.568\,05, 0.622\,96, 0.058\,49)$, and $(0.515\,76, 0.571\,23, 0.038\,69)$, for $b = 3$, 5, 7, and 9, respectively. The respective eigenvalues are 3, 6.607 7, 10.887 1, and 15.817 4.

References

[1] Elezović S, Knežević M and Milošević S 1987 *J. Phys. A: Math. Gen.* **20** 1215

[2] Milošecić and Živić I 1991 *J. Phys. A: Math. Gen.* **24** L833

[3] Stošić T, Stošić B, Milošević S and Stanley H E 1990 *Physica* **170A** 31

[4] Given J A and Mandelbrot B B 1983 *J. Phys. A: Math. Gen.* **16** L565

[5] Rammal R, Toulouse G and Vannimenus J 1984 *J. Physique* **45** 389

[6] Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062

[7] Dhar D 1988 *J. Physique* **49** 397